Modeling Financial Volatility with Multinomial Generalized Auto-Regressive Conditional Heteroskedasticity

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Abstract

Modeling the levels of volatility for a portfolio of assets has been imperative in analyzing individual financial decisions. In this paper, I explore models that allow for volatility clustering (ARCH and GARCH) and specifically look at a five-year dataset of the S&P 500 stocks to model the returns to a portfolio of assets using Multinomial GARCH (DCCM).

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1 Introduction

Understanding financial volatility plays an imperative role in the context of risk assessment and financial decision-making (portfolio and investment choices). Predicting and modeling volatility has, therefore, become extremely salient in financial econometrics.

Early models for financial time series, such as the Auto-Regressive (AR) and Moving Average (MA) models, assumed homoskedastic errors, where the variance remains constant over time. While these models provide a solid foundation, they fail to account for volatility clustering- the phenomenon that periods of high volatility are followed by high volatility and low volatility followed by low volatility. which is a common characteristic observed in financial returns.

To address this limitation, Engle (1982) introduced the Auto Regressive Conditional Heteroskedasticity (ARCH) model, which allows conditional variances to depend on past squared returns. This was a significant advancement as it enabled the modeling of volatility clustering. However, the ARCH model has its own limitations, including the need for a large number of parameters for long memory effects. Bollerslev (1986) extended the ARCH framework by proposing the Generalized ARCH (GARCH) model, which incorporates lagged conditional variances into the model specification, providing a more robust representation of volatility dynamics.

In this paper, I build on these foundational models by exploring Multinomial GARCH as a framework for modeling the volatility of a portfolio of assets. Using a five-year dataset of S&P 500 stocks, I aim to estimate and interpret the volatility dynamics within a multivariate framework, highlighting the benefits of modeling time-varying correlations across assets. This approach not only provides a comprehensive view of asset risk but also higlights the importance of asset interdependence in portfolio management.

2 Literature Review

In this section, I talk briefly about financial volatility in the context of stocks is typically defined as (and will be defined as for the course of this paper). I then explore how Univariate ARCH and GARCH model the volatility of stock returns, and further talk about current extensions in these models.

2.1 Financial Volatility

Financial volatility is, simply put, the variation in returns from a financial asset over a period of time measured by the standard deviation of returns. Asset volatility is an important part of the measurement of risk associated with the investment in a particular asset. If we consider a time series of returns for some asset $(r_t)_{t=1}^{N}$, we can think about the volatility of the asset as σ_t.

2.2 Univariate ARCH

Engle proposed that unobservable second moments can be modeled by specifying a functional form for the conditional variance and modeling first and second moments jointly. More simply, the argument is that if we are willing to make certain assumptions about the way variance behaves, we can predict volatility. Suppose we have a $N \times 1$ times series vector of returns, $(r_t)_{t=1}^{N}$. We can model the returns as a function of the mean returns μ plus some error term ϵ_t

$$r_t = \mu + \epsilon_t \tag{1}$$

Where the errors themselves can be modeled as,

$$\epsilon_t = e_t \sigma_t \tag{2}$$

a white noise process (e_t) times the volatility (σ_t). In this context, the variance in squared-returns is allowed to be conditional on past square returns, more formally, $\sigma_t^2 = Var[r_t^2|\mathcal{F}_{t-1}]$ where $\mathcal{F}_{t-1}$ is the sigma-algebra generated by $r_0, ..., r_{t-1}$. The ARCH model specifies the functional form of these conditional variances as follows,

$$\sigma_t^2 = \omega + \alpha_1 r_{t-1}^2 ... + \alpha_p r_{t-p}^2 \tag{3}$$

$$= \omega + \sum_{i=1}^{p} \alpha_i r_{t-i}^2 \tag{4}$$

In words, the ARCH model specifics the functional form of the variance in returns as a sum of p-lagged squared past squared returns.

2.3 Uni-variate GARCH

The Uni-variate Generalized Auto-regressive Conditional Heteroskedasticity model allowed for the conditional variances to be dependent on the conditional variances as well. Modeling variance as,

$$\sigma_t^2 = \omega + \alpha_1 r_{t-1}^2 ... + \alpha_p r_{t-p}^2 + \beta_1 \sigma_{t-1}^2 + ... + \beta_q \sigma_{t-q}^2 \tag{5}$$

$$= \omega + \sum_{i=1}^{p} \alpha_i r_{t-i}^2 + \sum_{i=1}^{q} \beta_i \sigma_{t-i}^2 \tag{6}$$

In practice, a GARCH(1,1) model does the best job of forecasting volatility (with the metric of bias-variance trade-off) and hence is extensively used in financial modeling.

2.4 Extensions

While the standard GARCH model is powerful, it doesn't always account for certain empirical phenomena like asymmetric responses to shocks (leverage effects). Several variants have been proposed to address these limitations:

1. EARCH (Exponential ARCH): Introduced by Nelson (1989), EARCH models impose

an autoregressive structure on the logarithm of the conditional variance, enabling it to take negative values and account for asymmetric effects.

2. QGARCH (Quadratic GARCH): Sentana (1995) proposed this model, which includes an additional term for errors, allowing for asymmetric reactions.

3. AGARCH (Asymmetric GARCH): Developed by Engle and Ng (1993), AGARCH models conditional variance as dependent on the squared and non-centralized errors, capturing the leverage effect.

4. TGARCH (Threshold GARCH): Zakoian (1994) introduced this model, where the conditional standard deviation depends on positive and negative parts of past returns separately, providing a closer alignment with empirical data.

3 Multinomial GARCH Model

The methods we have explored before are valuable in predicting the returns for individual assets, however, if we were interested in predicting the returns to a portfolio of stocks (allowing the individual stocks to be a GARCH process while allowing individual error terms to correlate with one and other) we would use a Multinomial GARCH model. Let returns from a stock i be a GARCH process, with N observations, formally, $(r_t)_{t=1}^{N}$. The Multinomial GARCH model proposes the following specification of errors from stock i at time t:

$$\epsilon_t = e_t H_t^{\frac{1}{2}} \tag{7}$$

Where H_t is a $N \times N$ matrix of conditional variances, $H^{1/2}$ is positive semi-definite and e_t is a $N \times 1$ vector of iid white noise, $e_t \sim (0, I_N)$. The parametrization for H_t as a function of $\mathcal{F}_{t-1}$ would allow each element to depend on q-lagged values of squared returns, p-lagged values of elements of H_t, and cross products of ϵ_t. As mentioned in the literature review, GARCH(1,1) is often used due to its ease and predictive power, it follows, for the multinomial GARCH case too we will consider a Multinomial GARCH(1,1) model.

3.1 Multinomial GARCH(1,1) - Dynamic Conditional Correlation Model (DCCM)

Like there are different ways to model the conditional variance (as in ARCH and GARCH) there are different ways to model the variance-covariance matrix H_t. For this empirical analysis, I will use the Dynamic Conditional Correlation (DCC) model as proposed by Engle (2001). The idea here is that H_t is constructed by the variances of uni-variate GARCH(1,1) process and the corelation between these individual processes. So, $H_t = \Delta_t C_t \Delta_t$. Here, $\Delta_t = diag(\sqrt{h_{11,t}}, ..., \sqrt{h_{NN,t}})$. h_t contains the standard deviations of individual uni-variate GARCH(1,1) process, and C_t is a time-varying correlational matrix (extended from constant correlation proposed by Bollerslev). The dynamics of C_t are modeled through an intermediate matrix Q_t given by,

$$Q_t = (1 - a - b)\bar{Q} + ae_{t-1}e_{t-1} + bQ_{t-1}.$$

$\bar{Q}$ is the unconditional covariance matrix of e_t, a and b are estimated parameters. Then C_t is derived by $C_t = diag(Q_t)^{-1/2} Q_t diag(Q_t)^{-1/2}$.

3.2 GMM Estimator

4 Data

To analyze and model stock returns using the Multinomial GARCH model, I use a dataset that consists of 505 stocks that were in the S&P 500 index between 2013 and 2018. The data comes from an available Kaggle dataset in conjunction with web scraping from Yahoo Finance. The data measures the opening price, closing price, day high, day low, and trade volume on a daily basis from February 8 2013 to February 7 2018. There are 619040 observations in this dataset. Below is a table of the summary of average prices for the top 10 most traded stocks (by average volume).

Table 1

	Name	Average Open	Average Close	Mean High	Mean Low
1	AAPL	109.01	109.03	109.91	108.1
2	AMD	5.6	5.6	5.71	5.48
3	BAC	17.7	17.69	17.86	17.53
4	CSCO	27.72	27.73	27.94	27.51
5	F	14.12	14.11	14.23	13.98
6	FB	96.39	96.41	97.32	95.4
7	GE	26.54	26.54	26.72	26.34
8	INTC	31.79	31.82	32.07	31.53
9	MSFT	51	51.03	51.4	50.6
10	MU	22.97	22.96	23.32	22.59

Since we are interested in looking at the returns from stocks, we use the price data to generate returns for each stock for each day. The return specification used is log returns which is defined as, $r_t = log(\frac{P_{t+1}}{P_t})$, where P_{t+1} is the price on the next day and P_t is the price today. Here is a graph of the average quarterly log returns (unweighted), this is what we would expect to see quarterly on average if we owned all 505 stocks.

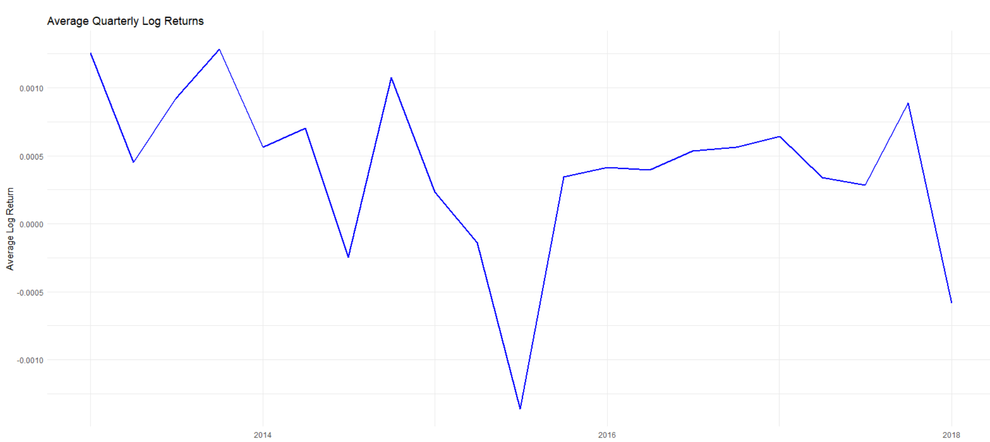

Figure 1: Average Quarterly Log Returns

5 Results

5.1 Volatility Clustering

The entire idea of modeling conditional heteroskedasticy is based on the strong prevalence of volatility clustering. In order to check whether there is indeed volatility clustering in our data we run a a few preliminary analysis.

One way we do this is by arranging the stocks in descending order based on the standard deviations of the returns and pull the 3 most volatile stocks. We find that the most volatile stocks in this dataset are CHK: Chesapeake Energy, AMD: Advanced Micro Devices, Inc., and FCX: Freeport-McMoRan Inc. Below is a picture of the stocks and their squared returns. The picture does indeed show that events of low returns are typically followed by low returns

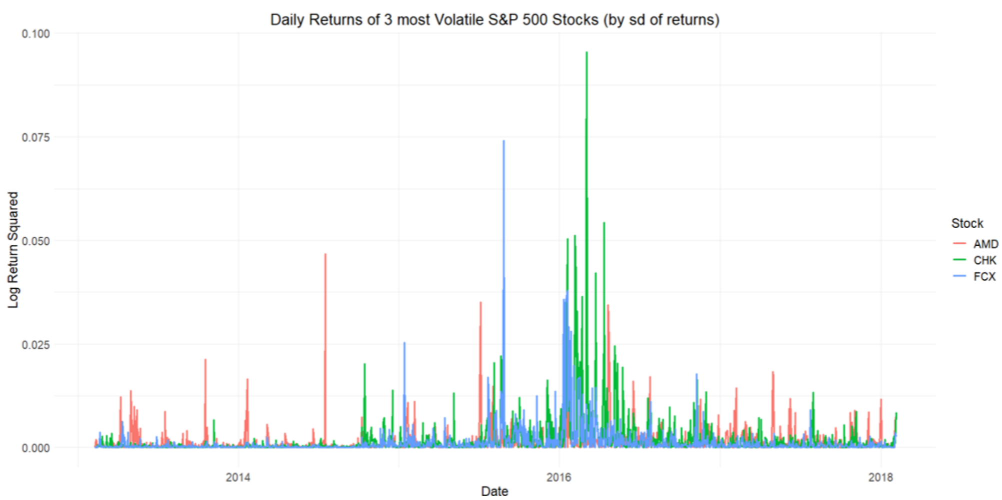

Figure 2: Volatility Clustering

and events of high returns are followed by high returns (prevalence of clustering). More formally, we can run an ARCH LM test to check whether the coefficient on the lagged squared return is significantly different than 0. Below is the result from using the ARCH model on CHK.

Table 2

	Dependent variable:
	Return Squared
Lagged Return Square	0.450***
	(0.025)
Constant	0.001***
	(0.0001)
Observations	1,257
R^2	0.203
Adjusted R^2	0.202
Residual Std. Error	0.005 (df = 1255)
F Statistic	318.668*** (df = 1; 1255)
Note:	*p<0.1; **p<0.05; ***p<0.01

Clearly, the lagged term is significantly different from 0. Another test we perform is the ARCH LM on the mean squared returns (across all 505 stocks), this test also yields evidence of volatility clustering.

Table 3

	Dependent variable:
	Average Return Squared
Lagged Average Return Squared	0.325***
	(0.027)
Constant	0.00004***
	(0.00001)
Observations	1,257
R^2	0.105
Adjusted R^2	0.104
Residual Std. Error	0.0002 (df = 1255)
F Statistic	146.915*** (df = 1; 1255)
Note:	*p<0.1; **p<0.05; ***p<0.01

5.2 S-GARCH(1,1)

Following the ARCH test, I also perform the Univariate GARCH(1,1) model using the traditional S-GARCH specification. Running the GARCH specification is important since in the multinomial model we treat individual stocks as univariate GARCH processes. Given that, I run an S-GARCH specification on CHK (the most volatile stock in this dataset) and find the results shown in the table below.

Table 4: GARCH(1,1) Model Coefficients

Parameter	Coefficient	Std..Error	z.Value	p.Value
mu	-0.0004	0.0007	-0.6213	0.5344
omega	0	0	1.8655	0.0621
alpha1	0.0893	0.0141	6.3301	0
beta1	0.9097	0.0128	71.2595	0

I also perform the GARCH model on the average returns for all the stocks and find the following coefficients.

Table 5: GARCH(1,1) Model Coefficients

Parameter	Coefficient	Std..Error	z.Value	p.Value
mu	0.0005	0.0002	2.8151	0.0049
omega	0	0	3.9816	0.0001
alpha1	0.2123	0.0228	9.3186	0
beta1	0.7377	0.0163	45.2577	0

5.3 Multinomial GARCH- DCCM

For this paper, I will be fitting the multinomial GARCH model to one portfolio of stocks (obviously this choice may be arbitrary and repeated across several different types and combinations of portfolios). The portfolio of stocks I will be choosing to model with the multinomial GARCH model is the top 10 most traded stocks (by volume), this should capture a good snippet of possible stocks in an average individuals' portfolio. I run the dynamic correlation specification, extensively explained in the multinmomial GARCH section.

I yield the 10-dimensional correlational and covariance matrices, along with the estimated coefficients of the GARCH model and proxy correlational matrix Q_t. The size of the table of coefficient summary makes it difficult to present results clearly, however, the analysis yields the following values for each stock. The time-dependent correlation matrix opens

```
    [AAPL].mu    [AAPL].ar1  [AAPL].omega [AAPL].alpha1  [AAPL].beta1      [AMD].mu     [AMD].ar1   [AMD].omega  [AMD].alpha1   [AMD].beta1      [BAC].mu     [BAC].ar1
[BAC].omega  [BAC].alpha1   [BAC].beta1     [CSCO].mu    [CSCO].ar1
 8.594244e-04 -6.304356e-02  1.403890e-04  2.859632e-01  1.904346e-01  5.558793e-04  2.975951e-02  2.847966e-04  1.872072e-01  5.982019e-01  1.083605e-03  2.055543e-02
3.352178e-05  1.269163e-01  7.439756e-01  5.212198e-04 -6.885079e-02
 [CSCO].omega [CSCO].alpha1  [CSCO].beta1       [FB].mu      [FB].ar1    [FB].omega  [FB].alpha1   [FB].beta1        [F].mu       [F].ar1    [F].omega   [F].alpha1
[F].beta1        [GE].mu       [GE].ar1    [GE].omega   [GE].alpha1
 1.203104e-06  9.405913e-04  9.920183e-01  1.251400e-03 -4.547372e-02  3.970049e-06  4.421907e-02  9.515584e-01 -1.334814e-04  6.100857e-02  8.948935e-05  3.376754e-01
3.025813e-01 -1.728283e-04 -4.464717e-02  1.303271e-05  1.407568e-01
   [GE].beta1     [INTC].mu    [INTC].ar1  [INTC].omega [INTC].alpha1  [INTC].beta1     [MSFT].mu    [MSFT].ar1  [MSFT].omega [MSFT].alpha1  [MSFT].beta1      [MU].mu
[MU].ar1    [MU].omega   [MU].alpha1    [MU].beta1  [Joint]dcca1
 7.831751e-01  6.583011e-04  2.777615e-02  5.817196e-05  2.147010e-01  4.768491e-01  1.031369e-03 -3.800208e-02  1.467585e-06  1.348815e-02  9.797223e-01  1.878551e-03
3.456478e-02  8.819875e-06  1.222168e-02  9.757706e-01  3.277104e-03
 [Joint]dccb1
 9.873211e-01
```

Figure 3: Coeffecients Multinomial GARCH

the opportunity for interesting exploratory analysis in stock interdependence. For example below is the time-dependent correlation between Apple and Microsoft, and Apple and GE.

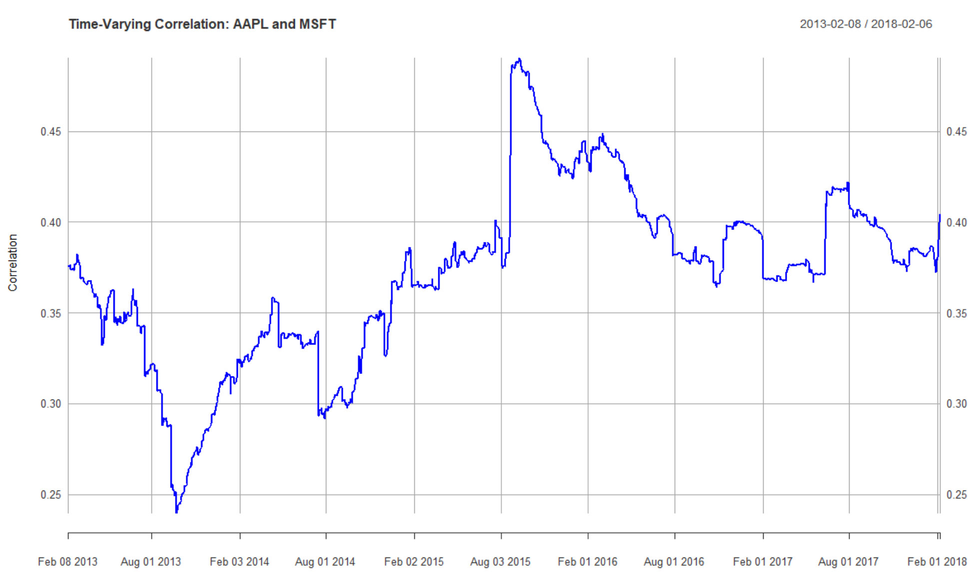

Figure 4: Apple Microsoft Time Dependent Correlation

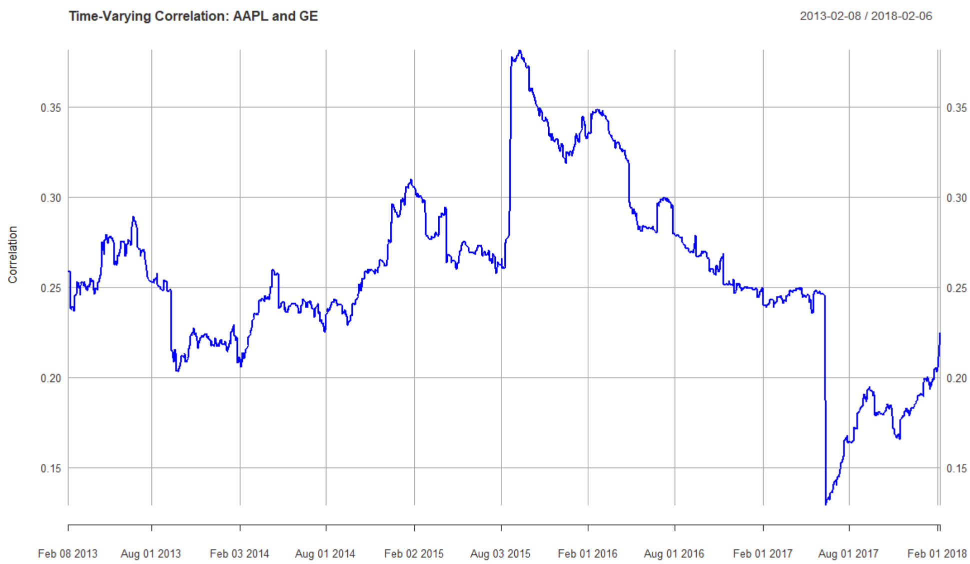

Figure 5: Apple GE Time Dependent Correlation

The multinomial GARCH model with dynamic correlation can also be used to forecast future correlation forecasts along with changes in returns given that independent stocks are allowed to correlate with one and other. For example below is the first correlational forecast based on the multinomial GARCH model. All of my analysis is performed in R-studio and

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]	[,7]	[,8]	[,9]	[,10]
[1,]	1.0000	0.2578	0.3398	0.3249	0.4113	0.3193	0.2314	0.3185	0.4154	0.3359
[2,]	0.2578	1.0000	0.2476	0.2241	0.2487	0.2213	0.1853	0.2286	0.2817	0.3718
[3,]	0.3398	0.2476	1.0000	0.4232	0.2552	0.4696	0.3658	0.3055	0.3294	0.2613
[4,]	0.3249	0.2241	0.4232	1.0000	0.2875	0.3916	0.3360	0.4151	0.4173	0.2824
[5,]	0.4113	0.2487	0.2552	0.2875	1.0000	0.2362	0.1814	0.2480	0.4065	0.3313
[6,]	0.3193	0.2213	0.4696	0.3916	0.2362	1.0000	0.3839	0.2884	0.3139	0.2729
[7,]	0.2314	0.1853	0.3658	0.3360	0.1814	0.3839	1.0000	0.1961	0.2383	0.1782
[8,]	0.3185	0.2286	0.3055	0.4151	0.2480	0.2884	0.1961	1.0000	0.4810	0.2941
[9,]	0.4154	0.2817	0.3294	0.4173	0.4065	0.3139	0.2383	0.4810	1.0000	0.2965
[10,]	0.3359	0.3718	0.2613	0.2824	0.3313	0.2729	0.1782	0.2941	0.2965	1.0000

Figure 6: First Period Correlation Forecasts

the code can be found in the appendix section of this paper.

6 Appendix

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